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## Path Integration on Homogeneous Spaces*)

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**Abstract:** The path integral for the free quantum motion on an arbitrary homogeneous space $\mathcal{M}$, which may be viewed as a group quotient $\mathcal{M} = G/H$, is considered. The group $G$ is the transformation group of $\mathcal{M}$ and $H$ is the stationary group of a fixed vector on $\mathcal{M}$. We expand the short time propagator in unitary irreducible representations of $G$. As the short time propagator is invariant under global transformations of $G$ and local transformations of $H$, only the so-called zonal spherical functions do contribute in the Fourier analysis. The path integral is performed explicitly by using the orthogonality of the representations. The correct normalized wave functions are given by associate spherical functions and the energy spectrum is obtained from the time derivative of the Fourier coefficients of the expansion. The general formulation will be applied to the following examples: (i) The non-relativistic or relativistic (Klein-Gordon) free particle in the $n$-dimensional flat space $R^n$. (ii) The propagation on spaces of constant positive or negative curvature.

## 1 Preliminaries

The path integral for the quantum propagator on symmetric spaces has attracted much attention in recent years. For example, Dowker [1] and Marinov and Terent'ev [2] have considered the dynamics on the manifold of simple compact Lie groups. Recently, the explicit path integral formulation on the groups $SU(2)$ and $SU(1,1)$ have become available [3,4]. These treatments are of great importance for the path integral solution of systems having the dynamical symmetry $SU(2)$ or $SU(1,1)$. The Feynman ansatz for these systems is changed into a path integral of a free particle moving on the dynamical group manifold. With the path integral realization of their dynamical symmetry many problems have become exactly path integrable. For a recent review on this work see [5].

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Other problems of interest are related to the quantum propagators on spaces with constant positive [3,6-11] or negative [7-9,12] curvature. In general the Lagrangian path integral on curved spaces acquires a non-trivial correction in the Lagrangian, which is proportional to the curvature [13,14]. However, in the case of constant curvature such a correction becomes trivial. It appears as an overall constant in the energy spectrum and has no physical effect. Indeed, these manifolds may be embedded into a flat space giving rise to a well-defined Feynman path integral [7].

In this work we consider the path integral on a homogeneous space $\mathcal{M}$, which may be viewed as a group quotient $\mathcal{M} = G/H$. Here $G$ is the transformation group on $\mathcal{M}$, i.e. it transforms any point on $\mathcal{M}$ into any other one, $\mathbf{x} \to \mathbf{x}' = g\mathbf{x}$, $g \in G$. $H$ is the stability group of a fixed point $\mathbf{a}$ on $\mathcal{M}$, $h\mathbf{a} = \mathbf{a}$, $h \in H$. The path integration is performed explicitly for an arbitrary homogeneous space. We obtain a general formula for the energy eigenvalues and also the normalized wave functions. As examples, we consider the free particle and the Klein-Gordon propagator in an $n$-dimensional flat space. The path integration on $n$-dimensional spaces of constant positiv or negative curvature is also discussed. For the latter the energy depended Green function is given in closed form.

## 2 The General Formulation

Our starting point is the path integral representation of the transition amplitude for a free quantum system on a homogeneous space $\mathcal{M}$ evolving from initial state $|\mathbf{x}_a\rangle$ to the final configuration $|\mathbf{x}_b\rangle$ in the time $T = t_b - t_a$. In the usual sliced time basis, $T = N\varepsilon$, this propagator is given by

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \lim_{N\to\infty} \int \cdots \int \prod_{i=1}^{N} K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) \prod_{i=1}^{N-1} d\mathbf{x}_j. \tag{2.1}$$

We have adopted the standard notation: $\mathbf{x}_j = \mathbf{x}(t_j)$, $\mathbf{x}_b = \mathbf{x}_N$ and $\mathbf{x}_a = \mathbf{x}_0$. $d\mathbf{x}$ is the Lebesgue measure on $\mathcal{M}$. For compact spaces $\int_{\mathcal{M}} d\mathbf{x} = |\mathcal{M}|$, where $|\mathcal{M}|$ is the volume of $\mathcal{M}$.

In Feynman's path integral approach the short time propagator is assumed to be given in the semiclassical form [15]

$$K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) = \sqrt{\det \left| \frac{i}{2\pi\hbar} \frac{\partial^2 S_j}{\partial \mathbf{x}_{j-1} \partial \mathbf{x}_j} \right|} \exp\{(i/\hbar) S_j\}, \tag{2.2}$$

where the classical action $S_j$ along the short time interval $\varepsilon$ is approximated by $[\Delta \mathbf{x}_j = \mathbf{x}_j - \mathbf{x}_{j-1},\ \bar{\mathbf{x}}_j = (\mathbf{x}_j + \mathbf{x}_{j-1})/2]$

$$S_j = \int_{t_{j-1}}^{t_j} L(\dot{x}, x)\, dt \approx L(\Delta \mathbf{x}_j/\varepsilon, \bar{\mathbf{x}}_j)\varepsilon. \tag{2.3}$$



In the Hamiltonian formulation of Feynman's path integral the short time propagator is taken to be the matrix element of the time evolution operator for an infinitesimal time interval $\varepsilon$:

$$K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) = \langle \mathbf{x}_j | e^{-(i/\hbar)H(\mathbf{p},\mathbf{x})\varepsilon} | \mathbf{x}_{j-1} \rangle \approx \langle \mathbf{x}_j | 1 - (i/\hbar) H(\mathbf{p}, \mathbf{x})\varepsilon\} | \mathbf{x}_{j-1} \rangle. \tag{2.4}$$

In ordinary non-relativistic quantum mechanics, where the Hamiltonian is given by $H(\mathbf{p}, \mathbf{x}) = \mathbf{p}^2/2m + V(\mathbf{x})$ , both approaches are known to be equivalent. On the other hand, the quantum Hamiltonian on a curved manifold $\mathcal{M}$ is essentially given by the Laplace-Beltrami operator $\Delta$ on $\mathcal{M}$, $H = -(\hbar^2/2m)\Delta + V(\mathbf{x})$. Here ordering problems arise in writing the Laplacian in terms of momentum operators [17]. In the Lagrangian path integral we do not have to deal with operators. However, the topology appears in additional terms to the Lagrangian, which are proportional to the curvature. In general there is no guarantee that both approaches give the same result. Only for spaces with constant curvature the corrections are trivial. Indeed, embedding such a manifold into a higher dimensional flat space and constraining by $\delta$-functions [3,7,8] yields the same propagator as the Hamiltonian formalism [10-12]. The only difference may be an overall additional constant in the energy spectrum.*) This difference may be due to an inappropriate choice of the Hamiltonian [10]. For example, Dowker [1] has shown that a conformally invariant operator on $\mathcal{M}$ is $\left(\Delta - \hbar^2 K/6\right)$ instead of the pure Laplace-Beltrami operator $\Delta$ ($K$ is the scalar curvature).

In the following discussion we will not restrict the short time propagator to be of the Lagrangian or Hamiltonian type. Our treatment will be applicable to both formulations. All we need to know is that the short time propagator for the free motion on an $n$-dimensional homogeneous manifold $\mathcal{M}$ is invariant under translations:

$$K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) = K(\mathbf{x}'_j, \mathbf{x}'_{j-1}, \varepsilon), \tag{2.5}$$

where

$$\mathbf{x}'_j = g\mathbf{x}_j,\ \forall j = 0, 1, \ldots, N. \tag{2.6}$$

$g$ is a group element of the transformation group $G$ acting on $\mathcal{M}$. Choosing a fixed point $\mathbf{a}$ on $\mathcal{M}$ an arbitrary $\mathbf{x}_j \in \mathcal{M}$ may be obtained through a local rotation:

$$\mathbf{x}_j = g_j \mathbf{a},\ g_j \in G. \tag{2.7}$$

The stability group $H$ is the subgroup of $G$ leaving the fixed vector $\mathbf{a}$ invariant:

$$h\mathbf{a} = \mathbf{a},\ \forall h \in H \subset G. \tag{2.8}$$

The homogeneous manifold $\mathcal{M}$ can be identified with the coset space $G/H$. With this construction, the short time propagator may be viewed as a function on the group manifold of $G$:

$$K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) = K(g_j, g_{j-1}, \varepsilon). \tag{2.9}$$

*) The authors of [11,12] believe that this is an essential difference. We do not think that an additional constant in the energy spectrum is of any physical significance. It only leads to a different definition of the ground state energy of the free system. The physics remains the same.

From the translation invariance follows that (2.9) is a function of the combination $g_{j-1}^{-1}g_j$ only:

$$K(g_j, g_{j-1}, \varepsilon) = K(gg_j, gg_{j-1}, \varepsilon) = K(g_{j-1}^{-1}g_j, \varepsilon). \tag{2.10}$$

On the other hand, multiplying any group element $g_j$ in (2.7) from the right with an arbitrary element of the stationary group $H$ gives rise to the same $\mathbf{x}_j$. Obviously, the short time propagator (2.10) is invariant with respect to left and right multiplication by an element of the subgroup $H$:

$$K(g, \varepsilon) = K(h_1^{-1}gh_2, \varepsilon),\ h_1, h_2 \in H. \tag{2.11}$$

Property (2.11) is the defining equation of the so-called zonal spherical functions [18] and may be considered as the matrix element of an unitary operator $\mathbf{K}(g, \varepsilon)$ acting in a Hilbert space $\mathcal{H}(G/H)$:*)

$$K(g, \varepsilon) = \langle \mathrm{a}|\mathbf{K}(g, \varepsilon)|\mathrm{a}\rangle. \tag{2.12}$$

Let us consider an unitary irreducible representation $\ell$ of the group $G$, which associates with each element $g \in G$ an unitary operator $\mathcal{D}^\ell(g)$ in a Hilbert space $\mathcal{H}^\ell$. Introducing a basis $\{\mathrm{b}_k\}$, $k = 0, 1, 2, \ldots, \dim\mathcal{H}^\ell - 1$, in this space we define the matrix elements of the representation $\ell$ by

$$\mathcal{D}^\ell_{mn}(g) = \langle \mathrm{b}_m|\mathcal{D}^\ell(g)|\mathrm{b}_n\rangle. \tag{2.13}$$

Furthermore, suppose that $\mathcal{H}^\ell$ contains a vector $|\mathrm{a}\rangle$ that is invariant under $H$, i.e. $\mathcal{D}^\ell(h)|\mathrm{a}\rangle = |\mathrm{a}\rangle$, $\forall h \in H$.†) Choosing the basis $\{\mathrm{b}_k\}$ in such a way that $|\mathrm{b}_0\rangle = |\mathrm{a}\rangle$, the matrix elements

$$\mathcal{D}^\ell_{00}(g) = \langle \mathrm{a}|\mathcal{D}^\ell(g)|\mathrm{a}\rangle \tag{2.14}$$

are the zonal spherical functions of the representation $\ell$. The notation is similar to that for the group $G = SO(3)$ where the zonal spherical functions are given by Legendre polynomials

$$\mathcal{D}^\ell_{00}(g) = P_\ell(\cos\theta).$$

The functions $\mathcal{D}^\ell_{00}(g)$ form a complete set for zonal spherical functions in the Hilbert space $\mathcal{H}(G/H) = \bigoplus_\ell \mathcal{H}^\ell$. In other words, any function $f(g)$, constant on the two-sided coset $H\backslash G/H$, can be decomposed in zonal spherical functions of unitary irreducible representations [19]:

$$f(g) = \sum\!\!\!\!\!\!\int_\ell d_\ell\, f_\ell\, \mathcal{D}^\ell_{00}(g), \tag{2.15}$$

$$f_\ell = \int_G f(g)\,\mathcal{D}^{\ell *}_{00}(g)\,dg. \tag{2.16}$$

*) Zonal spherical functions are eigenfunctions of the Laplace-Beltrami operator on $\mathcal{M} = G/H$.

†) It can be shown [16] that the subspace defined by this condition is one-dimensional, i.e. the vector $|\mathrm{a}\rangle$ is unique.



In the above $\sum\!\!\!\!\!\!\int_\ell$ stands for the orthogonal sum of all inequivalent unitary irreducible representations. The question, which representations have to be included in this sum, is in principle answered by the Frobenius theorem [20]. The number $d_\ell$ is defined by

$$\int_G \mathcal{D}^\ell_{00}(g)\mathcal{D}^{\ell' *}_{00}(g)\,dg = \frac{\delta(\ell', \ell)}{d_\ell}, \tag{2.17}$$

where $\delta(\ell', \ell) = \delta_{\ell'\ell}$ for discrete and $\delta(\ell', \ell) = \delta(\ell' - \ell)$ for continuous representation label ("angular momentum"), respectively. For compact groups $G$, $d_\ell$ is just the dimension of the corresponding representation. Unitary irreducible representations of non-compact groups are infinite dimensional. However, we may call $d_\ell$ "dimension" in this case, too.

After this excursion in the theory of group representations let us come back to the path integral. We have already mentioned, that the short time propagator (2.10) is a zonal spherical function. Consequently, it may be decomposed as

$$K(g_{j-1}^{-1}g_j, \varepsilon) = \sum\!\!\!\!\!\!\int_\ell d_\ell\, f_\ell(\varepsilon)\,\mathcal{D}^\ell_{00}(g_{j-1}^{-1}g_j). \tag{2.18}$$

The Fourier coefficients are given by

$$f_\ell(\varepsilon) = \int_G K(g, \varepsilon)\mathcal{D}^{\ell *}_{00}(g)\,dg. \tag{2.19}$$

Futhermore, through relation (2.7) we may identify the coordinates of $\mathbf{x}_j$ with those parameters of the group element $g_j$, which do not belong to the subgroup $H$. Therefore, the volume element $d\mathbf{x}_j$ appearing in the path integral (2.1), may be changed into the normalized Haar measure of the group $G$ by multiplication with the identity $1 = \int_H dh$.

$$d\mathbf{x}_j = |\mathcal{M}| \int_H dg_j \tag{2.20}$$

The volume $|\mathcal{M}|$ shows up because of the relation $|\mathcal{M}| = \int_\mathcal{M} d\mathbf{x} = |\mathcal{M}| \int_G dg$. Formally, it may be viewed as the "Jacobian" $\partial(\mathbf{x})/\partial(g) = |\mathcal{M}|$.

Inserting everything into the path integral we obtain

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \lim_{N\to\infty} \int \ldots \int \prod_{j=1}^N \left\{ \sum\!\!\!\!\!\!\int_{\ell_j} d_{\ell_j}\, f_{\ell_j}(\varepsilon)\,\mathcal{D}^{\ell_j}_{00}(g_{j-1}^{-1}g_j) \right\} \prod_{j=1}^{N-1} |\mathcal{M}|\,dg_j. \tag{2.21}$$

Making use of the orthogonality relation

$$\int_G \mathcal{D}^{\ell_j}_{00}(g_{j-1}^{-1}g_j)\mathcal{D}^{\ell_{j+1}}_{00}(g_j^{-1}g_{j+1})\,dg_j = \frac{\delta(\ell_j, \ell_{j+1})}{d_{\ell_j}}\,\mathcal{D}^{\ell_j}_{00}(g_{j-1}^{-1}g_{j+1}), \tag{2.22}$$

the integrations in (2.21) may easily be performed. The result is

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \sum\!\!\!\!\!\!\int_\ell \left\{ \lim_{N\to\infty} [f_\ell(\varepsilon)|\mathcal{M}|]^N \right\} \frac{d_\ell}{|\mathcal{M}|}\mathcal{D}^\ell_{00}(g_0^{-1}g_N). \tag{2.23}$$

Using the group property $\mathcal{D}^{\ell}_{00}(g_0^{-1}g_N) = \sum_m \mathcal{D}^{\ell}_{m0}(g_N)\mathcal{D}^{\ell *}_{m0}(g_0)$, the propagator may be written in the standard form

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \sum\!\!\!\!\!\!\int_{\ell} \exp\{-(i/\hbar)E_\ell T\} \sum_m \Psi_{\ell m}(\mathbf{x}_b)\Psi^*_{\ell m}(\mathbf{x}_a), \tag{2.24}$$

where

$$E_\ell = (i\hbar/T)\log\left\{\lim_{N\to\infty} [f_\ell(T/N)|\mathcal{M}|]^N\right\}, \tag{2.25}$$

$$\Psi_{\ell m}(\mathbf{x}) = \sqrt{d_\ell/|\mathcal{M}|}\,\mathcal{D}^{\ell}_{m0}(g) \tag{2.26}$$

are the energy spectrum and normalized wave functions, respectively. The matrix elements $\mathcal{D}^{\ell}_{m0}(g)$ are called associate spherical functions [18,19]. They are eigenfunctions of the Laplace-Beltrami operator on $\mathcal{M} = G/H$. The limit in (2.25) can be calculated by Taylor expansion:

$$\lim_{N\to\infty} [|\mathcal{M}|f_\ell(T/N)]^N = \lim_{N\to\infty}\left[|\mathcal{M}|f_\ell(0)\left(1+\frac{\dot f_\ell(0)}{f_\ell(0)}\frac{T}{N}\right)\right]^N = \exp\left\{\frac{\dot f_\ell(0)}{f_\ell(0)}T\right\}, \tag{2.27}$$

where $\dot f_\ell(\varepsilon) = \frac{d}{d\varepsilon}f_\ell(\varepsilon)$. In the last step we have made use of the normalization $K(\mathbf{x},\mathbf{a},0) = \delta(\mathbf{x}-\mathbf{a}) = \delta(g)/|\mathcal{M}|$, which yields $f_\ell(0) = 1/|\mathcal{M}|$. Hence, the energy spectrum is given by the time derivative of the Fourier coefficents at $\varepsilon = 0$:

$$E_\ell = i\hbar|\mathcal{M}|\dot f_\ell(0). \tag{2.28}$$

The correctness of this relation is easily shown in the Hamiltonian form of the short time propagator:

$$K(g,\varepsilon) = \langle \mathbf{x}|1-(i/\hbar)H(\mathbf{p},\mathbf{x})\varepsilon|\mathbf{a}\rangle.$$

Inserting this expression into eq. (2.19) we find

$$\begin{aligned}
\dot f_\ell(0) &= \frac{1}{i\hbar}\int_G \langle\mathbf{x}|H(\mathbf{p},\mathbf{x})|\mathbf{a}\rangle \mathcal{D}^{\ell *}_{00}(g)\,dg \\
&= \frac{1}{i\hbar}\sum\!\!\!\!\!\!\int_{\ell' m'}\int_G \langle\mathbf{x}|H(\mathbf{p},\mathbf{x})|\Psi_{\ell' m'}\rangle\langle\Psi_{\ell' m'}|\mathbf{a}\rangle \mathcal{D}^{\ell *}_{00}(g)\,dg \\
&= \frac{1}{i\hbar}\sum\!\!\!\!\!\!\int_{\ell' m'} E_{\ell'}\int_G \langle\mathbf{x}|\Psi_{\ell' m'}\rangle\langle\Psi_{\ell' m'}|\mathbf{a}\rangle \mathcal{D}^{\ell *}_{00}(g)\,dg.
\end{aligned}$$

Using the explicit form (2.26) of the wave functions $\Psi_{\ell m}(\mathbf{x}) = \langle\mathbf{x}|\Psi_{\ell m}\rangle$ and the relation $\mathcal{D}^{\ell}_{m0}(h) = \delta_{m0}$, $h \in H \subset G$, the result is

$$\dot f_\ell(0) = \frac{1}{i\hbar}\sum\!\!\!\!\!\!\int_{\ell' m'} E_{\ell'}\frac{d_{\ell'}}{|\mathcal{M}|}\mathcal{D}^{\ell' *}_{m'0}(h)\int_G \mathcal{D}^{\ell'}_{m'0}(g)\mathcal{D}^{\ell *}_{00}(g)\,dg = \frac{E_\ell}{i\hbar|\mathcal{M}|}.$$



Finally we would like to mention, that if the Fourier coefficients are of the type $f_\ell(\varepsilon) = |\mathcal{M}|^{-1}\exp\{-(i/\hbar)\varepsilon E_\ell\}$, the short time propagator and the finite time propagator are of the same form. If in this case the semiclassical approximation for the short time propagator is exact then it is also exact for the finite time propagator.

Another point worth to mention is the following. In eq. (2.23) we see that the propagator is a zonal spherical function depending on $g_0^{-1}g_N$. It is known, that zonal functions on rank one spaces depend only on one variable. This parameter may be identified with the geodesic distance $s = d(\mathbf{x}_b, \mathbf{x}_a)$ between the initial and final positions on $\mathcal{M}$. Consequently, the transition amplitude on a homogeneous manifold is only a function of the geodesic distance between $\mathbf{x}_a$ and $\mathbf{x}_b$:

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = K(s, T). \tag{2.29}$$

## 3 Path Integration in the Euclidean Space

As a first application we will consider the path integral in an $n$-dimensional flat space $R^n$. The corresponding transformation group is the Euclidean group. It is the semi-direct product of translations and rotations in $n$ dimensions, $G = T^n \circledS SO(n)$, acting via the map:

$$g:\ \mathbf{x} \longmapsto h\mathbf{x} + \mathbf{r}, \quad g \in G. \tag{3.1}$$

In the above $h$ is an $n\times n$ matrix representation of the subgroup $H = SO(n)$. The parameters of the group element $g = g(\mathbf{r}, h)$ are the $n(n-1)/2$ Euler angles of $h$ and the $n$ coordinates of the translation vector $\mathbf{r}$ given (for convenience) in polar coordinates $(r, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$. The group composition law is

$$g(\mathbf{r}_1, h_1)g(\mathbf{r}_2, h_2) = g(\mathbf{r}_1 + h_1\mathbf{r}_2, h_1h_2). \tag{3.2}$$

An arbitrary group element may be decomposed into a translation and a rotation [19]:

$$g(\mathbf{r}, h) = g(\mathbf{r}, \mathbf{1})g(\mathbf{o}, h) = g(\mathbf{o}, h)g(h^{-1}\mathbf{r}, \mathbf{1}), \tag{3.3}$$

where $\mathbf{1}$ stands for the $n\times n$ unit matrix and $\mathbf{o}$ is the $n$-dimensional null vector. Obviously, any point $\mathbf{r}$ in $R^n$ may be obtained via a translation of the origin $\mathbf{o}$:

$$g:\ \mathbf{o} \longmapsto \mathbf{r}. \tag{3.4}$$

Accordingly, we may restrict $g$ in (3.3) to the form $g(\mathbf{r}, \mathbf{1})$ as the origin is invariant under rotations $g(\mathbf{o}, h)$. This corresponds to the choice of the fixed vector $\mathbf{a}$ to be the origin.

The zonal spherical functions $D^k_{00}(g)$ are given by Bessel functions [19]:

$$D^k_{00}(g) = \Gamma(n/2)(2/kr)^{(n-2)/2}J_{(n-2)/2}(kr), \quad k \in [0, \infty), \tag{3.5}$$

where $r$ is the radial polar coordinate of the translation vector $\mathbf{r}$ in $g(\mathbf{r}, h)$. The basis states are usually labelled by $k$, $j$ and $M$, corresponding to the conserved linear momentum ($|\mathbf{p}| = \hbar k$), angular momentum $j$, and its degeneracy $M$, respectively. Note, that we use the standard label $k$ instead of $\ell$ for the representations.

With the general formulation of section 2 the Fourier decomposition on the Euclidean group $T^n \otimes SO(n)$ for rotational invariant functions is:*)

$$f(g) = \int_0^\infty f_k\, d_k\, D^k_{00}(g)\, dk, \quad f_k = \int_G f(g)\, D^{k*}_{00}(g)\, dg. \tag{3.6}$$

In the above $dg$ is the invariant volume element given by

$$dg = dr dh, \tag{3.7}$$

where $dh$ is the normalized Haar measure of $H = SO(n)$, $\int_H dh = 1$, and $dr$ is the invariant element of $T^n$, which coincides with the Lebesgue measure on $R^n$. This means, $|\mathcal{M}| = 1$.

The "dimension" $d_k$ is found to be [21]:

$$d_k = k^{n-1}/[2^{n-1}\pi^{n/2}\Gamma(n/2)]. \tag{3.8}$$

### 3.1 The Non-Relativistic Free Particle

Firstly, we consider the non-relativistic free particle in $n$ dimensions, whose short time propagator is given by

$$K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2} \exp\{(i/\hbar)S_j\}, \quad S_j = \frac{m}{2\varepsilon}(\Delta\mathbf{x}_j)^2. \tag{3.9}$$

Let us consider the group element $g_j = g(\mathbf{x}_j, 1)$. Obviously, the origin is mapped onto $\mathbf{x}_j$ via the translation $g_j$. An investigation of the combination

$$g_{j-1}^{-1} g_j = g^{-1}(\mathbf{x}_{j-1}, 1) g(\mathbf{x}_j, 1) = g(\mathbf{x}_j - \mathbf{x}_{j-1}, 1) \tag{3.10}$$

shows that this is the translation mapping $\mathbf{o}$ onto $\Delta\mathbf{x}_j$ . Having this in mind the short time propagator in (3.9) is a function $f(g_{j-1}^{-1} g_j)$ on $G$ which depends only on the parameter $r = |\Delta\mathbf{x}_j|$ of the group element (3.10). Therefore, the Fourier decomposition (3.6) may be applied to $(z/\pi)^{n/2}\exp\{(i/\hbar)S_j\}$ ($z = m/2i\hbar\varepsilon$) and the corresponding coefficient is given by

$$f_k(\varepsilon) = 2z^{n/2}(2/k)^{(n-2)/2} \int_0^\infty dr\, r^{n/2} e^{-zr^2} J_{(n-2)/2}(kr). \tag{3.11}$$

In (3.11) the integration over the subgroup $H$ and the group parameters $(\varphi_1, \ldots, \varphi_{n-1})$ of the translation vector $\mathbf{r}$ has been performed. Using the integral formula (**Re** $\alpha > 0$, $\beta > 0$, **Re** $\nu > -1$):

$$\int_0^\infty dx\, x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) = \beta^\nu (2\alpha)^{-\nu-1} \exp\{-\beta^2/4\alpha\},$$

*) This integral transformation is known as Hankel transformation.



we find

$$f_k(\varepsilon) = \exp\left\{-\frac{i\hbar k^2}{2m}\varepsilon\right\}. \tag{3.12}$$

To be more explicit we have the decomposition

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2} \exp\left\{\frac{im}{\hbar 2\varepsilon}|\Delta\mathbf{r}_j|^2\right\} = \int_0^\infty dk\, \exp\left\{-\frac{i\hbar k^2\varepsilon}{2m}\right\} d_k\, D^k_{00}(g_{j-1}^{-1} g_j). \tag{3.13}$$

With the aid of the orthogonality (2.17) the path integration can be performed leading to the following integral representation of the free particle propagator:

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \int_0^\infty dk\, \exp\left\{-\frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}T\right\} d_k\, D^k_{00}(g_a^{-1} g_b). \tag{3.14}$$

The energy spectrum is read off from (3.12), $E_k = \hbar^2 k^2/2m$. The normalized wave functions are [21]

$$\Psi_{kjM}(\mathbf{x}) = i^j (k/r^{n-2})^{1/2} J_{j+(n-2)/2}(kr) \sqrt{\Gamma(n/2)/2\pi^{n/2}}\, Y_{jM}(\mathbf{e}). \tag{3.15}$$

$Y_{jM}(\mathbf{e})$ are the hyperspherical harmonics in $n$ dimensions [7], where $\mathbf{e} = \mathbf{x}/r$. The integers $m_i$ of the set $M = (m_1, \ldots, m_{n-2})$ are related by $j \ge m_1 \ge m_2 \ge \ldots \ge m_{n-3} \ge |m_{n-2}| \ge 0$. As the Fourier coefficient is an exponential with exponent linear in time, the short time and finite time propagator have the same form:

$$K(\mathbf{x}_b, \mathbf{x}_a, T) = \left(\frac{m}{2\pi i\hbar T}\right)^{n/2} \exp\left\{\frac{im}{2\hbar T}|\mathbf{x}_b - \mathbf{x}_a|^2\right\}. \tag{3.16}$$

### 3.2 The Klein-Gordon Propagator

Secondly, we perform the path integral for the relativistic spinless free particle in $n$ dimensions described by the Hamiltonian $H = c\sqrt{\mathbf{p}^2 + m^2c^2}$. In this case we will take the Hamiltonian form of the short time propagator. Following the nice calculation of Fukutaka and Kashiwa [10] we find

$$\begin{aligned} K(\mathbf{x}_j, \mathbf{x}_{j-1}, \varepsilon) &= \langle \mathbf{x}_j | \exp\left\{-(i/\hbar)\varepsilon c\sqrt{\mathbf{p}^2 + m^2c^2}\right\} | \mathbf{x}_{j-1}\rangle \\ &= \int d\mathbf{p}_j (2\pi\hbar)^{-n} \exp\left\{(i/\hbar)\left(\mathbf{p}_j \cdot \Delta\mathbf{x}_j - c\varepsilon\sqrt{\mathbf{p}_j^2 + m^2c^2}\right)\right\} \\ &= 2ic\varepsilon\left(\frac{mc}{2\pi i\hbar}\right)^{(n+1)/2} \frac{K_{(n+1)/2}\left((imc/\hbar)\sqrt{c^2\varepsilon^2 - (\Delta\mathbf{x}_j)^2}\right)}{[c^2\varepsilon^2 - (\Delta\mathbf{x}_j)^2]^{(n+1)/4}}. \end{aligned} \tag{3.17}$$

In the above $K_\nu(z)$ is the modified Bessel function of the third kind. The short time propagator (3.17) does not look at all like a Feynman amplitude, that is, being of the form $\exp\{(i/\hbar)S_j\}$, where $S_j = -mc^2\varepsilon\sqrt{1 - (\Delta\mathbf{x}_j)^2/c^2\varepsilon^2}$.*) This point has

*) The Lagrangian corresponding to $H = c\sqrt{\mathbf{p}^2 + m^2c^2}$ is $L = -mc^2\sqrt{1 - \dot{\mathbf{x}}^2/c^2}$.

led Mizrahi [17] to the conclusion that the path integral (2.1) with (3.17) does not tend to a well-defined functional in the limit $N \to \infty$. Indeed, he has used the Klein-Gordon system as a counterexample which is not path integrable. However, in the following we will explicitly perform the path integral.

Firstly, we calculate the Fourier coefficent by making use of the integral formula:

$$\int_0^\infty dr\ r^{\nu+1}(r^2+\beta^2)^{-\nu/2-3/4}K_{\nu+3/2}\left(a\sqrt{r^2+\beta^2}\right)J_\nu(kr) = \\ = \sqrt{\frac{\pi}{2\beta^2}}\,a^{-\nu-3/2}k^\nu \exp\left\{-\beta\sqrt{a^2+k^2}\right\}.$$

The result is

$$f_k(\varepsilon) = \exp\left\{-\frac{i}{\hbar}\varepsilon c\sqrt{m^2c^2+\hbar^2k^2}\right\}. \tag{3.18}$$

We immediately obtain the energy spectrum $E_k = c(m^2c^2+\hbar^2k^2)^{1/2}$. The wave functions are the same as in (3.15). Again $f_k(\varepsilon)$ is an exponential with exponent linear in $\varepsilon$. The finite time propagator is of the same form as the short time propagator:

$$K(\mathbf{x}_b,\mathbf{x}_a,T) = 2icT\left(\frac{m\gamma}{2\pi i\hbar T}\right)^{(n+1)/2} K_{(n+1)/2}(imc^2T/\hbar\gamma). \tag{3.19}$$

Here we have set $\gamma = [1-s^2/c^2T^2]^{-1/2} = [1-v^2/c^2]^{-1/2}$, where $s = |\mathbf{x}_b - \mathbf{x}_a|$ is the geodesic distance between $\mathbf{x}_a$ and $\mathbf{x}_b$. Writing the Bessel function as a hypergeometric series,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\,e^{-z}\ {}_2F_0(1/2-\nu,1/2+\nu;-1/2z), \tag{3.20}$$

we obtain the interesting representation:

$$K(\mathbf{x}_b,\mathbf{x}_a,T) = \gamma\left(\frac{m\gamma}{2\pi i\hbar T}\right)^{n/2}\exp\left\{\frac{i}{\hbar}S_{cl}\right\}\ {}_2F_0(-n/2,(n+2)/2;i\hbar\gamma/2mc^2T), \tag{3.21}$$

where $S_{cl} = -mc^2T/\gamma$ is the classical action. The propagator is of the Feynman type, $K = Ae^{(i/\hbar)S_{cl}}$. However, the prefactor $A$ cannot be obtained from the VanVleck-Pauli determinant (2.2).*)

It is remarkable that for the unphysical values $n = -2$ and 0 the hypergeometric function in (3.21) becomes unity and therefore the semiclassical approximation $(\hbar \to 0)$

$$K_{sc}(\mathbf{x}_b,\mathbf{x}_a,T) = \gamma\left(\frac{m\gamma}{2\pi i\hbar T}\right)^{n/2}\exp\left\{\frac{i}{\hbar}S_{cl}\right\} \tag{3.22}$$

is found to be exact.

Formula (3.21) explicitly demonstrates the equivalence of the classical $(\hbar \to 0)$, large time $(T \to \infty)$ and non-relativistic limit $(c \to \infty)$.

*) Note that $(\partial^2 S_{cl}/\partial x_a^k \partial x_b^l) = -(m\gamma/T)[\delta_{kl} + (\gamma^2/c^2T^2)(x_b^k - x_a^k)(x_b^l - x_a^l)]$. The calculation of the determinant in eq. (2.2) gives the semiclassical approximation (3.22).



# 4 Path Integration on Uniformly Curved Spaces

In the following we consider the Feynman path integral of a free particle moving in a constantly curved space of constant positive or negative curvature. Embedding this space in a higher dimensional flat space, we write down a well-defined Lagrangian path integral. Then, in the integration procedure we constrain all paths to lie on the curved subspace. We will neglect the curvature correction as it gives rise to a constant phase factor in the propagator only. Instead, we will fix the energy scale such that the ground state energy is set to zero, $E_0 = 0$.

## 4.1 The Path Integral on $S^n = SO(n+1)/SO(n)$

First, we consider the quantum motion on an $n$-dimensional manifold with constant positive curvature $K = 1/R^2$. The line element $ds$ of this space is given in polar coordinates by

$$ds^2 = \left(1 - r^2/R^2\right)^{-1}dr^2 + r^2d\Omega^2. \tag{4.1}$$

$\Omega$ stands for an unit vector in $n$ dimensions. Introducing a new variable $\chi$ by $\sin\chi = r/R,\ \chi \in [0,\pi]$, it can also be put in the form [22]

$$ds^2 = R^2d\chi^2 + R^2\sin^2\chi d\Omega^2. \tag{4.2}$$

This looks like the line element of an $(n+1)$-dimensional flat space in polar coordinates $(R,\chi,\Omega)$ with $dR = 0$. Therefore, we may achieve the embedding by

$$x^o = R\cos\chi,\ \mathbf{x} = R\sin\chi\,\Omega \implies ds^2 = dx^{o2} + d\mathbf{x}^2. \tag{4.3}$$

The $n$-dimensional space of constant positive curvature $K = 1/R^2$ is equivalent to the sphere $S^n$ of radius $R$ in the $(n+1)$-dimensional Euclidean space. The transformation group on this surface is the rotation group $G = SO(n+1)$. The stationary group of, e.g., the north pole $a^\mu = (1,0,\dots,0)$ is $H = SO(n)$. Group theoretical we have $S^n = SO(n+1)/SO(n)$. Note that the north pole corresponds to the origin in the curved space, $\mathbf{a} = \mathbf{o}$.

We begin with the short time propagator in the embedding space:

$$K(\Delta s_j,\varepsilon) = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{(n+1)/2}\exp\left\{\frac{im}{\hbar 2\varepsilon}\left[\Delta x_j^{o2} + \Delta\mathbf{x}_j^2\right]\right\} \\ = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{(n+1)/2}\exp\left\{(im/\hbar 2\varepsilon)\left[\Delta R_j^2 + 2R_{j-1}R_j - 2x_j^\mu x_{j-1\,\mu}\right]\right\}. \tag{4.4}$$

The volume element is $dx^\mu = R^n\sin^{n-1}\chi\,dRd\chi d\Omega$ and $\Delta s_j = d(\mathbf{x}_j,\mathbf{x}_{j-1})$. The constraint $dR = 0$ may be put into the path integral by the substitution

$$(m/2\pi i\hbar\varepsilon)^{1/2}\exp\left\{(im/\hbar 2\varepsilon)\Delta R_j^2\right\} \implies \delta(R_j - R_{j-1})\exp\{(i/\hbar)\varepsilon V_c\} \tag{4.5}$$

with $R_a = R_b = R$.*) $V_c$ is the constant correction due to the curvature and will be determined by fixing the energy scale to $E_0 = 0$.

The Feynman ansatz on $S^n = SO(n+1)/SO(n)$ follows to be

$$K(s,T) = R^{-n} \exp\{(i/\hbar)TV_c\} \lim_{N\to\infty} \int \cdots \int \prod_{j=1}^{N} K(e_j^\mu, e_{j-1}^\mu, \varepsilon) \prod_{j=1}^{N-1} de_j^\mu, \qquad (4.6)$$

where the short time propagator reads

$$K(e_j^\mu, e_{j-1}^\mu, \varepsilon) = \left(\frac{mR^2}{2\pi i\hbar\varepsilon}\right)^{n/2} \exp\left\{\frac{imR^2}{\hbar\varepsilon}\left(1 - e_j^\mu e_{j-1\,\mu}\right)\right\}. \qquad (4.7)$$

$e^\mu = x^\mu/R$ is an unit vector on $S^n$ and $de^\mu$ the corresponding volume element. According to section 2 we may change the integration into one over the rotation group $SO(n+1)$:

$$de_j^\mu = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}} \int_{H=SO(n)} dg_j. \qquad (4.8)$$

Note that the volume of $S^n$ is $|S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$.

The zonal spherical functions are given in terms of Gegenbauer polynomials [19]:

$$\mathcal{D}_{00}^\ell(g) = \frac{(n-2)!\,\ell!}{(\ell+n-2)!} C_\ell^{(n-1)/2}(\cos\theta), \ \ \ell = 0,1,2,\ldots, \qquad (4.9)$$

where $\theta$ is the polar angle between $e^\mu = ga^\mu$ and the north pole, $\cos\theta = e^\mu a_\mu$.†) The dimension of the representation $\ell$ is $d_\ell = (2\ell+n-1)(\ell+n-2)!/[\ell!\,(n-1)!]$.

The Fourier coefficient of the short time propagator (4.7) has been given in terms of modified Bessel functions [7]:

$$f_\ell(\varepsilon) = \frac{1}{|S^n|}\left(\frac{2\pi mR^2}{i\hbar\varepsilon}\right)^{1/2} \exp\left\{\frac{imR^2}{\hbar\varepsilon}\right\} I_{\ell+(n-1)/2}(mR^2/i\hbar\varepsilon). \qquad (4.10)$$

For small $\varepsilon$ we have

$$f_\ell(\varepsilon) = \frac{1}{|S^n|}\left[1 - \frac{i\hbar\varepsilon}{2mR^2}\left(\left(\ell + \frac{n-1}{2}\right)^2 - 1/4\right) + O(\varepsilon^2)\right]. \qquad (4.11)$$

One easily obtains $f_\ell(0) = 1/|S^n|$ and $\dot{f}_\ell(0) = [\ell(\ell+n-1)+n(n-2)/4]\hbar/|S^n|2imR^2$. With $V_c = n(n-2)\hbar^2/8mR^2$ we find the energy spectrum and the normalized wave functions:

$$E_\ell = \ell(\ell+n-1)\hbar^2/2mR^2, \qquad (4.12)$$

$$\Psi_{\ell M}(\mathbf{x}) = \left[\frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\right]^{1/2} Y_{\ell M}(e^\mu). \qquad (4.13)$$

*) The remaining $\delta(R_b - R_a)$ is set to unity. For more details see [3,8]

†) The geodesic distance $s = d(\mathbf{x}, \mathbf{a})$ is $s = R\theta$



The hyperspherical harmonics are the same as in eq. (3.15) with $n \to n+1$. The complete propagator reads

$$\begin{aligned} K(s,T) &= \frac{1}{R^n|S^n|}\sum_{\ell=0}^{\infty} e^{-(i/\hbar)E_\ell T}\, d_\ell\, \mathcal{D}_{00}^\ell(g_0^{-1}g_N) \\ &= \frac{1}{R^n|S^n|}\sum_{\ell=0}^{\infty} e^{-(i/\hbar)E_\ell T}\, \frac{2\ell+n-1}{n-1}\, C_\ell^{(n-1)/2}\big(\cos(s/R)\big), \end{aligned} \qquad (4.14)$$

where $s = R\arccos(e_b^\mu e_{a\mu})$ is the geodesic distance between initial and final position.

## 4.2 The Propagator on $\Lambda^n = SO(n,1)/SO(n)$

Our last example will be the Lagrangian path integral on a uniformly curved space with negative curvature $K = -1/R^2$. The corresponding line element is

$$ds^2 = \left(1 + r^2/R^2\right)^{-1} dr^2 + r^2 d\Omega^2. \qquad (4.15)$$

Again we introduce a new variable $\chi$ by $\sinh\chi = r/R$, $\chi \in [0,\infty)$ and put the line element in the form

$$ds^2 = R^2 d\chi^2 + R^2 \sinh^2\chi\, d\Omega^2. \qquad (4.16)$$

$\Omega$ stands for an $n$-dimensional unit vector as before. This geometry can be embedded in an $(n+1)$-dimensional flat Minkowski space by setting [22]:

$$x^o = R\cosh\chi,\ \mathbf{x} = R\sinh\chi\,\Omega \implies ds^2 = -dx^{o\,2} + d\mathbf{x}^2. \qquad (4.17)$$

We may identify the embedded surface with a "time-like" hyperboloid*) in the Minkowski space. Note that $SO(n,1)$ is transformation group on this surface. The stability group of the origin $a^\mu = (1,0,\ldots,0)$ is $H = SO(n)$. Group theoretical the negatively curved space, denoted by $\Lambda^n$, may be identified with the coset space $SO(n,1)/SO(n)$.

The short time propagator in this space reads [7,8]:

$$\begin{aligned} K(\Delta s_j, \varepsilon) &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2}\left(\frac{mi}{2\pi\hbar\varepsilon}\right)^{1/2} \exp\left\{\frac{im}{\hbar 2\varepsilon}\left[-\Delta x_j^{o\,2} + \Delta \mathbf{x}_j^2\right]\right\} \\ &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2}\left(\frac{mi}{2\pi\hbar\varepsilon}\right)^{1/2} \exp\left\{-\frac{im}{\hbar 2m}\left[\Delta R_j^2 + 2R_jR_{j-1} - 2x_{j-1}^\mu x_{j\,\mu}\right]\right\}. \end{aligned} \qquad (4.18)$$

The metric is $g^{\mu\nu} = diag(+,-,\ldots,-)$ and the volume element is given in polar coordinates by $dx^\mu = R^n \sinh^{n-1}\chi\, dRd\chi d\Omega$.

Constraining on the surface $R_j = R$ as before:

$$(mi/2\pi\hbar\varepsilon)^{1/2} \exp\left\{-(im/2\varepsilon)\Delta R_j^2\right\} \implies \delta(R_j - R_{j-1}) \exp\{(i/\hbar)\varepsilon V_c\}, \qquad (4.19)$$

*) It is the upper sheet of the two-sheeted hyperbola: $x^{o\,2} - \mathbf{x}^2 = R^2 > 0$, $x^o > 0$.

we obtain the Feynman ansatz for the propagator on the pseudosphere $\Lambda^n$:

$$K(s,T) = R^{-n}\exp\{(i/\hbar)TV_c\}\lim_{N\to\infty}\int\prod_{j=1}^{N}K(e_j^\mu, e_{j-1}^\mu,\varepsilon)\prod_{j=1}^{N-1}de_j^\mu,$$
$$K(e_j^\mu, e^\mu,\varepsilon) = \left(\frac{mR^2}{2\pi i\hbar\varepsilon}\right)^{n/2}\exp\left\{-\frac{imR^2}{\hbar\varepsilon}\left(1-e_j^\mu e_{j-1\,\mu}\right)\right\}. \quad (4.20)$$

$e^\mu = x^\mu/R$ is an unit vector in the Minkowski space lying on $\Lambda^n$. In changing the path integration into group integrals we have to know the invariant Haar measure of $SO(n,1)$. This may be found from the one of $SO(n+1)$ by analytical continuation in the group parameters. The volume of $\Lambda^n$ is infinite, but the "Jacobian" appears to be $|\Lambda^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, the same as for $S^n$.

The zonal sperical functions are given by Gegenbauer functions:

$$\mathcal{D}_{00}^\ell(g) = \frac{(n-2)!\,\Gamma(\ell+1)}{\Gamma(\ell+n-1)}\,C_\ell^{(n-1)/2}(\cosh\theta), \quad (4.21)$$

where $\theta$ is the "angle" between $e^\mu = ga^\mu$ and $a^\mu$, $\cosh\theta = e^\mu a_\mu$. The geodesic distance $s = d(e^\mu, a^\mu)$ is $s = \theta R$.

For the Fourier decomposition of the short time propagtor only the continuous fundamental series $\ell = -(n-1)/2 + i\rho$, $\rho \geq 0$ of $SO(n,1)$ does contribute [19], where the "dimension" is given by $d_\ell = 2|\Gamma((n-1)/2+i\rho)|^2/[|\Gamma(i\rho)|^2\Gamma(n)]$. Here the decomposition reduces to the generalized Mehler transformation. Indeed, following our suggestion [7] the generalized Mehler transformation has been utilized by Grosche and Steiner [12] for the path integration on $\Lambda^n$ without explicitly refering to the group theoretical connections. The Fourier coefficient of (4.20) has been obtained in [7]:

$$f_\ell(\varepsilon) = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\left(\frac{2mR^2}{\pi i\hbar\varepsilon}\right)^{1/2}\exp\left\{-\frac{imR^2}{\hbar\varepsilon}\right\}K_{i\rho}(mR^2/i\hbar\varepsilon)$$
$$= \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\left(1+\frac{\rho^2+1/4}{2imR^2}\hbar\varepsilon + O(\varepsilon^2)\right). \quad (4.22)$$

As expected, we have $f_\ell(0) = 1/|\Lambda^n|$. The time derivative of (4.22) at $\varepsilon = 0$ is $\dot f_\ell(0) = (\rho^2+1/4)\hbar/2im|\Lambda^n|R^2$. With $V_c = \hbar^2/8mR^2$ the energy spectrum is that of a free particle having momentum $|\mathbf{p}| = \hbar\rho$:

$$E_\rho = \hbar^2\rho^2/2mR^2. \quad (4.23)$$

The normalized wave functions are

$$\Psi_{\rho kM}(\mathbf{x}) = Z_{\rho k}(\chi)\left[\Gamma(n/2)/2\pi^{n/2}\right]Y_{kM}(\mathbf{e}), \quad (4.24)$$

where $Y_{kM}(\mathbf{e})$ are the hyperspherical harmonics in $n$ dimensions and

$$Z_{\rho k}(\chi) = [\Gamma((n-1)/2+k+i\rho)/\Gamma(i\rho)]\,\sinh^{(2-n)/2}\chi\,P_{-1/2+i\rho}^{(2-n)/2}(\cosh\chi). \quad (4.25)$$



Recall that $r = R\sinh\chi$. The functions $Z_{\rho k}(\chi)$ have first been introduced in [7] and differ from that of Bander and Itzykson [23] in the normalization factor. They obey the orthogonality relation*)

$$\int_0^\infty Z_{\rho k}(\chi)Z_{\rho' k}^*(\chi)\sinh^{n-2}\chi\,d\chi = \delta(\rho-\rho'). \quad (4.26)$$

We obtain an integral representation for the propagator:

$$K(s,T) = R^{-n}|\Lambda^n|^{-1}\int_0^\infty d\rho\exp\left\{-\frac{i}{\hbar}E_\rho T\right\}d_\ell\,\mathcal{D}_{00}^\ell(g_0^{-1}g_N)$$
$$= \frac{\Gamma(\frac{n+1}{2})(n-2)!}{R^n 2\pi^{(n+1)/2}}\int_0^\infty d\rho\frac{\Gamma\left(\frac{3-n}{2}+i\rho\right)}{\Gamma\left(\frac{n-1}{2}+i\rho\right)}\exp\left\{-\frac{i\hbar T\rho^2}{2mR^2}\right\}d_\ell\,C_{-\frac{n-1}{2}+i\rho}^{\frac{n-1}{2}}\left(\cosh(s/R)\right), \quad (4.27)$$

where $s = R\,\mathrm{arccosh}(e_b^\mu e_{a\mu})$ is the geodesic distance. From this, the energy dependent Green function can be obtained in closed form. A detailed calculation, given in the appendix, yields:†)

$$G(s,E) = \frac{me^{2\pi i\epsilon}}{\pi\hbar^2}\left(\frac{-1}{2\pi R^2\sinh(s/R)}\right)^{(n-2)/2}Q_{-1/2-i\nu}^{(n-2)/2}\left(\cosh(s/R)\right), \quad (4.28)$$

where $\epsilon = 0\,(1/2)$ for $n$ odd (even) and $\nu = (2mR^2E/\hbar^2)^{1/2}$. $Q_\mu^q(z)$ is the Legendre function of the second kind. The propagator can be put in the form:

$$K(s,T) = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2}\left(\frac{-1}{2\pi R\sinh(s/R)}\frac{d}{ds}\right)^{(n-1)/2}\exp\{(i/\hbar)S_{cl}\}, \quad (4.29)$$

for $n$ odd, and

$$K(s,T) = \sqrt{2}R\left(\frac{m}{2\pi i\hbar T}\right)^{3/2}\left(\frac{-1}{2\pi R\sinh(s/R)}\frac{d}{ds}\right)^{(n-2)/2}\int_{s/R}^\infty\frac{t\exp\{imR^2t^2/2\hbar T\}}{\sqrt{\cosh t-\cosh(s/R)}}dt, \quad (4.30)$$

for $n$ even. $S_{cl} = ms^2/2T$ is the classical action. The cases with $n=2$ and 3 have already been obtained by Gutzwiller [24]. For $n=1$ and $n=3$ the semiclassical approximation is exact. Taking the flat space limit $R\to\infty$ the propagator reduces to that of the usual $n$-dimensional free particle (3.16). The same is true for the Green function. See the appendix for the flat space limits.

*) This is easily obtained from the orthogonality relations for the group representations. Note the misprint in ref. [7].

†) The Green function given by Grosche and Steiner [12] is wrong as it is obtained through the unphysical regularization $E\to E-i\delta$, $\delta>0$.

## 5 Final Remarks

In this paper we have presented an exact path integral treatment on arbitrary homogenous spaces. The path integration can be reduced to the application of orthogonality relations of group representations. The normalized wave functions are matrix elements of unitary irreducible representations. They are the associate spherical functions which are eigenfunctions of the Laplace-Beltrami operator in this space. The energy spectrum is obtained from the Fourier coefficient of the short time propagator.

However, the present formalism is also applicable to the path integral on group spaces itself. In this case the stability group reduces to the identidy $H = \{e\}$. Here the zonal spherical functions are given by the characters of unitary irreducible representations [18], $\chi^\ell(g) = \mathrm{Tr}\,\mathcal{D}^\ell(g)$, and the normalized wave functions follow to be,

$$\Psi_{\ell mn}(\mathbf{x}) = \sqrt{\frac{d_\ell}{|\mathcal{M}|}}\,\mathcal{D}^\ell_{mn}(g). \tag{5.1}$$

Therefore, our treatment also includes the formulation for the path integration on group manifolds as a special case. The quantum propagator on group spaces has already been discussed by Dowker [1]. For a discussion of the functional integral on unitary groups see also the work by Marinov and Terent'ev [2].

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## Appendix: Green Function and Propagator on a Negatively Curved Space

The derivation of the Green function and the propagator will be similar to that of Grosche and Steiner [12]. However, these authors have implicitly used the regularization $E \to E - i\delta$, $\delta > 0$, for the calculation of the Green function, which has led them to a wrong result.

We will begin with the integral representation for the propagator given in eq. (4.27). With $d_\ell = 2|\Gamma((n-1)/2 + i\rho)|^2/[|\Gamma(i\rho)|^2\Gamma(n)]$, $|\Gamma(i\rho)|^2 = \pi/\rho\sinh\pi\rho$ and the relation

$$\Gamma\left(\frac{n-1}{2} - i\rho\right)\Gamma\left(1 - \frac{n-1}{2} + i\rho\right) = \pi/\sin\left(\pi\frac{n-1}{2} - i\pi\rho\right),$$



we find for the propagator:

$$K(s,T) = \frac{(-1)^k\Gamma((n-1)/2)}{R^n 2\pi^{n/2}}\int_0^\infty d\rho\,\frac{\rho\sinh(\pi\rho)}{\sin[\pi(\epsilon - i\rho)]}\,C^{k+\epsilon}_{-(k+\epsilon)+i\rho}\big(\cosh(s/R)\big)\,e^{-a\rho^2}, \tag{A1}$$

where $a = i\hbar T/2mR^2$ and $k = \left[\frac{n-1}{2}\right]$ is the integer part of $(n-1)/2$. In other words, $(n-1)/2 = k + \epsilon$ where $\epsilon = 0\,(1/2)$ for $n$ odd (even). Using the relation

$$C^{k+\epsilon}_{-(k+\epsilon)+i\rho}(z) = \frac{\Gamma(\epsilon)}{2^k\Gamma(k+\epsilon)}\frac{d^k}{dz^k}C^\epsilon_{-\epsilon+i\rho}(z)$$

we find

$$K(s,T) = \left(\frac{-1}{2\pi R^2}\frac{d}{d\cosh(s/R)}\right)^k K_\epsilon(s,T), \tag{A2}$$

where

$$K_\epsilon(s,T) = \frac{-1}{2\pi R^{1+2\epsilon}}\int_0^\infty d\rho\,\frac{\rho\sinh(\pi\rho)}{\sin[\pi(i\rho - \epsilon)]}\frac{\Gamma(\epsilon)}{\pi^\epsilon}C^\epsilon_{-\epsilon+i\rho}\big(\cosh(s/R)\big)\,e^{-a\rho^2}. \tag{A3}$$

With the following relations:

$$\sin(i\pi\rho) = i\sinh(\pi\rho), \qquad \sin[\pi(i\rho - 1/2)] = -\cosh(\pi\rho),$$

$$\lim_{\epsilon\to 0}\Gamma(\epsilon)\,C^\epsilon_{i\rho}\big(\cosh(s/R)\big) = (2/i\rho)\cos(\rho s/R),$$

$$\left[\Gamma(1/2)/\sqrt{\pi}\right]\,C^{1/2}_{-1/2+i\rho}(\cosh(s/R)) = P_{-1/2+i\rho}\big(\cosh(s/R)\big),$$

the $\epsilon$-propagator can be put into the form

$$\begin{aligned} K_0(s,T) &= (R\pi)^{-1}\int_0^\infty d\rho\,\cos(\rho s/R)\,e^{-i\hbar T\rho^2/2mR^2}, \\ K_{1/2}(s,T) &= (2\pi R^2)^{-1}\int_0^\infty d\rho\,\rho\tanh(\pi\rho)\,P_{-1/2+i\rho}\big(\cosh(s/R)\big)\,e^{-i\hbar T\rho^2/2mR^2}. \end{aligned} \tag{A4}$$

For $\epsilon = 0$ the propagator can directly be calculated using [25]*)

$$\int_0^\infty e^{-\beta x^2}\cos(bx)\,dx = \sqrt{\pi/4\beta}\,\exp\{-b^2/4\beta\}, \quad \mathrm{Re}\beta > 0.$$

We obtain the result

$$K_0(s,T) = \sqrt{\frac{m}{2\pi i\hbar T}}\exp\{(i/\hbar)(ms^2/2T)\}. \tag{A5}$$

The propagator in two dimensions cannot be found in closed form. However, we may express it in simple functions using the integral representation of the Legendre function of the first kind [23]

$$\tanh(\pi\rho)\,P_{-1/2+i\rho}\big(\cosh(s/R)\big) = \frac{\sqrt{2}}{\pi}\int_{s/R}^\infty \frac{\sin\rho t}{\sqrt{\cosh t - \cosh(s/R)}}\,dt,$$

*) The integrability condition is taken care of by the regularization $m \to m + i\eta$, $\eta > 0$ [7,26].

and the formula

$$\int_0^\infty \rho \sin(\rho t) e^{-a\rho^2}\, d\rho = \sqrt{\pi/a}\,(t/4a)\, e^{-t^2/4a}.$$

We find

$$K_{1/2}(s,R) = \sqrt{2}\, R \left(\frac{m}{2\pi i\hbar T}\right)^{3/2} \int_{s/R}^\infty \frac{t \exp\{-mR^2t^2/2i\hbar T\}}{\sqrt{\cosh t - \cosh(s/R)}}\, dt. \qquad \text{(A6)}$$

Thus, we have obtained eq. (4.30).

The energy dependent Green function is the Fourier transform of the propagator:*)

$$G(s,E) = \frac{1}{i\hbar}\int_0^\infty K(s,T) \exp\{(i/\hbar)(E+i\delta)T\}\, dT, \quad \delta > 0. \qquad \text{(A7)}$$

With relation (A2) we have

$$G(s,E) = \left(\frac{-1}{2\pi R^2}\frac{d}{d\cosh(s/R)}\right)^k G_\epsilon(s,E), \qquad \text{(A8)}$$

and $G_\epsilon(s,E)$ follows from eq. (A4):

$$\begin{aligned} G_0(s,E) &= \frac{2mR}{\pi(i\hbar)^2}\int_0^\infty d\rho\, \frac{\cos(\rho s/R)}{\rho^2+\gamma^2}, \\ G_{1/2}(s,E) &= \frac{m}{\pi(i\hbar)^2}\int_0^\infty d\rho\, \frac{\rho\tanh(\pi\rho)}{\rho^2+\gamma^2}\, P_{-1/2+i\rho}\big(\cosh(s/R)\big). \end{aligned} \qquad \text{(A9)}$$

In the above we have set $\gamma^2 = -(2mR^2/\hbar^2)(E+i\delta)$. Note that $\gamma$ is only unique up to a $\pm$ sign: $\gamma = \pm i\nu(1+i\delta/E)^{1/2}$, where $\nu = \sqrt{2mR^2E/\hbar^2}$. Roots are to be understood as positive roots. The integrals (A9) are exactly calculable [25]:

$$\int_0^\infty \frac{\cos(bx)}{x^2+\gamma^2}\, dx = (\pi/2\gamma)\, e^{-\gamma b}, \quad b>0, \quad \mathbf{Re}\ \gamma > 0, \qquad \text{(A10)}$$

$$\int_0^\infty \frac{x\tanh(\pi x)}{x^2+\gamma^2}\, P_{-1/2+ix}(\cosh b)\, dx = Q_{-1/2+\gamma}(\cosh b), \quad \mathbf{Re}\ \gamma > 0. \qquad \text{(A11)}$$

In both formulas we have the integrability condition $\mathbf{Re}\ \gamma > 0$, which means that we can apply (A10) and (A11) only if we choose the minus sign for $\gamma$ as $\delta > 0$:

$$\gamma = -i\nu, \quad \nu = \sqrt{2mR^2E/\hbar^2}. \qquad \text{(A12)}$$

The authors of ref. [12] have taken the plus sign in the application of (A11). This implies the regularization $E \to E - i\delta$, $\delta > 0$, which does not yield a physical result. In their derivation of the odd-dimensional Green function, which is different from ours, they have made a similar mistake. Their formula (V.12) is only valid for $\mathbf{Re}\ \sqrt{\beta\gamma} > 0$, where they have defined $\beta = mR^2r^2/2i$ and $\gamma = -iE$, giving $\sqrt{\beta\gamma} = \pm i\sqrt{mR^2r^2E/2}$. If we set $E \to E + i\delta$ the minus sign has to be taken in order to have $\mathbf{Re}\ \sqrt{\beta\gamma} > 0$. Grosche and Steiner [12] have taken the plus sign.

*) This differs from the definition of ref. [12] by a minus sign.



With the correct choice of the sign as in (A12) we obtain for the Green functions:

$$\begin{aligned} G_0(s,E) &= \frac{m}{\pi\hbar^2}\sqrt{-2\pi R^2}\, \sinh^{1/2}(s/R)\, Q^{-1/2}_{-1/2-i\nu}\big(\cosh(s/R)\big), \\ G_{1/2}(s,E) &= \frac{m}{\pi(i\hbar)^2}\, Q_{-1/2-i\nu}\big(\cosh(s/R)\big). \end{aligned} \qquad \text{(A13)}$$

In the last step we have made use of the relation

$$\exp\{i\nu s/R\} = \sqrt{2/\pi}\,\nu\, \sinh^{1/2}(s/R)\, Q^{-1/2}_{-1/2-i\nu}\big(\cosh(s/R)\big).$$

Finally, with the formulas

$$\begin{aligned} \frac{d^k}{dz^k}\left[(z^2-1)^{1/4} Q_\mu^{-1/2}(z)\right] &= (z^2-1)^{1/4-k/2} Q_\mu^{k-1/2}(z), \\ \frac{d^k}{dz^k} Q_\mu(z) &= (z^2-1)^{-k/2} Q_\mu^k(z), \end{aligned} \qquad \text{(A14)}$$

we arrive at eq. (4.28). The first formula is easily obtained by induction using the relation $(z^2-1)^{1/2}(d/dz)Q_\mu^{k-1/2}(z) = Q_\mu^{k+1/2}(z) + (k-1/2)(z/\sqrt{z^2-1})Q_\mu^{k-1/2}(z)$. The second one is a standard formula for the Legendre functions of the second kind.

The flat space limit $R \to \infty$ of the Green function can be calculated with the aid of the limiting relation for Legendre functions [27]:

$$\lim_{\mu\to\infty} (e^{-i\pi}/\mu)^q\, Q_\mu^q\big(\cosh(z/\mu)\big) = K_q(z).$$

$K_q(z)$ is the modified Bessel function of the third kind. Performing the limit yields

$$\lim_{R\to\infty} G(s,E) = e^{i\pi(2\epsilon-1)}\frac{2}{i\hbar}\left(\frac{m}{2\pi i\hbar}\right)^{n/2}\left(\frac{2E}{ms^2}\right)^{(n-2)/4} K_{(n-2)/2}\left(\sqrt{2mE}\,(s/i\hbar)\right), \qquad \text{(A15)}$$

which coincides, up to the phase factor, with the Green function obtained from the free propagator (3.16).

For the odd-dimensional ($\epsilon = 0$) propagator the flat space limit is easily taken:

$$\begin{aligned} \lim_{R\to\infty} K(s,T) &= \sqrt{\frac{m}{2\pi i\hbar T}}\lim_{R\to\infty}\left(\frac{-1}{2\pi R\sinh(s/R)}\frac{d}{ds}\right)^{(n-1)/2} \exp\{ims^2/\hbar 2T\} \\ &= \sqrt{\frac{m}{2\pi i\hbar T}}\left(\frac{-1}{2\pi s}\frac{d}{ds}\right)^{(n-1)/2} \exp\{ims^2/\hbar 2T\} \\ &= \left(\frac{m}{2\pi i\hbar T}\right)^{n/2} \exp\{ims^2/\hbar 2T\}. \end{aligned}$$

In the case of even dimension ($\epsilon = 1/2$) we first consider the expression

$$\lim_{R\to\infty} \sqrt{2}\,R \int_{s/R}^{\infty} \frac{t\,\exp\{imR^2t^2/2\hbar T\}}{\sqrt{\cosh t - \cosh(s/R)}}\,dt = \lim_{R\to\infty} \sqrt{2}\,R \int_{s/R}^{\infty} \frac{t\,\exp\{imR^2t^2/2\hbar T\}}{(R/\sqrt{2})\sqrt{t^2R^2 - s^2}}\,dt.$$

The substitution $u = t^2R^2 - s^2$, $du = 2tdt$, then leads to

$$\exp\{ims^2/2\hbar T\} \int_0^{\infty} u^{-1/2} \exp\{imu/2\hbar T\}\,du = (2\pi i\hbar T/m)^{1/2} \exp\{(i/\hbar)(ms^2/2T)\}.$$

Inserting this into the propagator for $\epsilon = 1/2$ gives the correct flat space limit:

$$\lim_{R\to\infty} K(s,T) = \frac{m}{2\pi i\hbar T} \left(\frac{-1}{2\pi s}\frac{d}{ds}\right)^{(n-2)/2} \exp\{ims^2/\hbar 2T\}$$
$$= \left(\frac{m}{2\pi i\hbar T}\right)^{n/2} \exp\{ims^2/\hbar 2T\}.$$

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